Vibrations

By the end of this lesson, you should be able to:

- *Describe the motion over time of vibrating systems*
- *Understand the natural frequencies of systems*
- *Be able to calculate resonant frequencies*
- *Have a better appreciation for ODEs ☺*

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Outline

- Conservative Systems (Mass + Spring)
- Damped Vibrations (Mass + Spring + Resistance)
- Forced Vibrations

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Vibrations

- Anytime you have a spring and a mass, you get vibrations
- You can think of vibrations as oscillations between kinetic (mass) and potential (spring) energy
- These vibrations may be damped by resistance
- We use Differential Equations to solve for the motion of vibrating systems over time
- We can then use them to tune the values of stiffness, mass, and resistance to amplify (ie, harmonics) or attenuate vibrations

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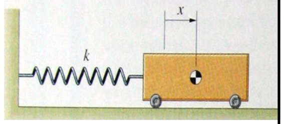
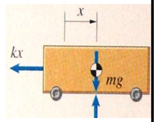
Conservative Systems





1) Draw Free-Body Diagram

2) Sum Forces $\Sigma F_x = -kx = ma$

3) Change into ODE $-kx = m\dfrac{d^2x}{dt^2}$

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

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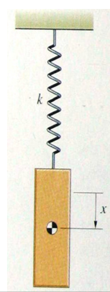
Another Conservative System

2) Sum Forces $\Sigma F_y = mg - ky = ma$

3) Change into ODE $mg - ky = m\dfrac{d^2y}{dt^2}$

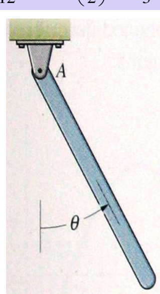
$$\frac{d^2y}{dt^2} + \frac{k}{m}y = g$$



A third Conservative System

2) Sum Moments $\Sigma M_A = -mg\sin\theta = I\alpha$

$$I_A = \frac{1}{12}ml^2 + m\left(\frac{l}{2}\right)^2 = \frac{ml^2}{3}$$



3) Change into ODE $-mg\dfrac{l}{2}\sin\theta = \dfrac{ml^2}{3}\dfrac{d^2\theta}{dt^2}$

$$\frac{d^2\theta}{dt^2} + \frac{3g}{2l}\sin\theta = 0$$

3) For small θ $\dfrac{d^2\theta}{dt^2} + \dfrac{3g}{2l}\theta = 0$

Solutions to Homogeneous ODEs

$$\frac{d^2x}{dt^2} + \omega^2 = 0$$

In the examples above:

$$\omega^2 = \frac{k}{m} \ , \ \frac{3g}{2l}$$

$$x = A\sin\omega t + B\cos\omega t \quad = \quad E\sin(\omega t - \phi)$$

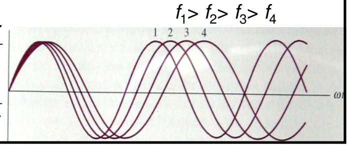
$$A = E\cos\phi, \quad B = -E\sin\phi$$

$\tau_1 < \tau_2 < \tau_3 < \tau_4$

$f_1 > f_2 > f_3 > f_4$

Period $\tau = \dfrac{2\pi}{\omega}$

Natural Frequency $f = \dfrac{\omega}{2\pi}$

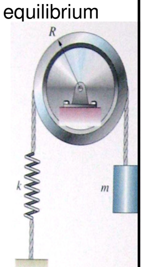


Example

The pulley has radius R and moment of inertia I, and the cable does not slip relative to the pulley. The mass m is displaced downward a distance h from its equilibrium position and released from rest at t=0.

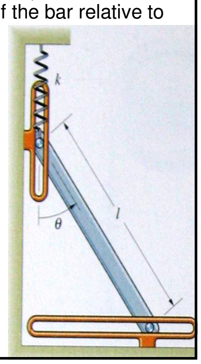
What is the natural frequency of the resulting vibrations?

Determine the position of the mass relative to its equilibrium position as a function of time



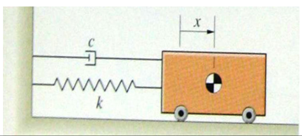
Example

The spring attached to the slender bar of mass m is unstretched when θ=0. Neglecting friction, determine the natural frequency of small vibrations of the bar relative to its equilibrium position



Damped Motion

- All real motion encounters resistance
- Resistance is good! It prevents vibrations
- As engineers, we often want to find the “optimum” resistance – just enough to prevent vibrations, without slowing us down too much.
- This optimal damping is called *Critical Damping*.
  Not enough is called *Subcritical Damping*.
  Too much is called *Supercritical Damping*.



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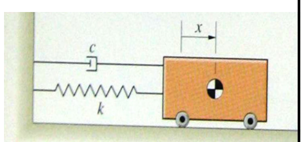
Damped Vibrations



$$-c\frac{dx}{dt}-kx=m\frac{d^2x}{dt^2}$$

$$\omega^2=\frac{k}{m} \quad , \quad \varsigma=\frac{c}{2m}$$

$$\frac{d^2x}{dt^2}+2\varsigma\frac{dx}{dt}+\omega^2x=0$$

Subcritical Damping: $\varsigma < \omega$

Critical Damping: $\varsigma = \omega$

Supercritical Damping: $\varsigma > \omega$

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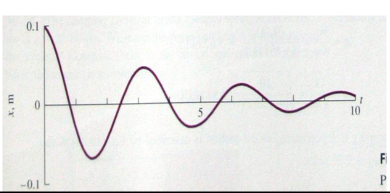
Damped Vibrations

- Amplitude Decreases over time
- Frequency is slower than undamped: $\omega_d=\sqrt{\omega^2-d^2}$
- Frequency remains constant – it doesn’t get slower over time; only amplitude changes over time



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Damped Vibrations

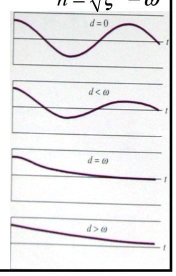
$$\frac{d^2x}{dt^2}+2\varsigma\frac{dx}{dt}+\omega^2 x=0$$

Subcritical Damping: $\varsigma < \omega$ $\quad x=e^{-\varsigma t}\left(A\sin\omega_d t+B\cos\omega_d t\right)$

Supercritical Damping: $\varsigma > \omega$ $\quad x=Ce^{-(\varsigma-h)t}+De^{-(\varsigma+h)t}$ $\quad h=\sqrt{\varsigma^2-\omega^2}$

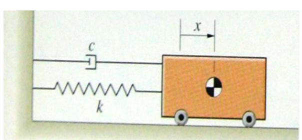
Critical Damping: $\varsigma = \omega$ $\quad x=Ce^{-\varsigma t}+Dte^{-\varsigma t}$

(h=0)



Example

The damped spring-mass oscillator has mass m = 2 k, spring constant k= 8 N/m, and damping constant c = 1 N-s/m. At t = 0, the mass is released from rest in the position x = 0.1 m.
Determine its position as a function of time.



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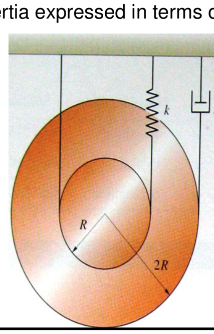
Example

The 40 N stepped disk is released from rest with the spring unstretched. Determine the position of the center of the disk as a function of time if R = 1 m, k = 10 N/m, c = 4 N-s/m, and the moment of inertia expressed in terms of the mass m of the disk is I = $3mR^2$.



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Forced Vibrations

A motor, or a person, can vibrate at a given frequency. These vibrations interact with the natural vibrations of the system. If they happen to be at the same frequency, resonance can occur, amplifying the vibrations.

$$\frac{d^2x}{dt^2} + 2\varsigma\frac{dx}{dt} + \omega^2 x = a(t)$$

Mathematically, forced vibrations do not produce homogeneous solutions – a particular solution must be added to the homogeneous solution: $x = x_h + x_p$

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Sinusoidal Forces

$$\frac{d^2x}{dt^2} + 2\varsigma\frac{dx}{dt} + \omega^2 x = a(t) \qquad a(t) = a_0 \sin\omega_0 t + b_0 \cos\omega_0 t$$

$$x_p = A_p \sin\omega_0 t + B_p \cos\omega_0 t$$

$$A_p = \frac{\left(\omega^2 - \omega_0^2\right)a_0 + 2\varsigma\omega_0 b_0}{\left(\omega^2 - \omega_0^2\right)^2 + 4\varsigma^2\omega_0^2}$$

$$B_p = \frac{\left(\omega^2 - \omega_0^2\right)b_0 - 2\varsigma\omega_0 a_0}{\left(\omega^2 - \omega_0^2\right)^2 + 4\varsigma^2\omega_0^2}$$

The system approaches the steady-state solution over time as the natural system dampens out

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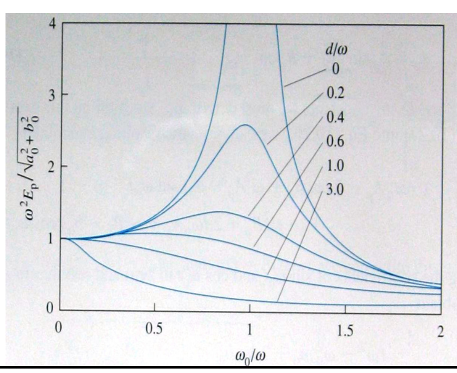
Resonance



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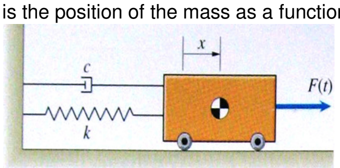
Example

An engineer designing a vibration isolation system for an instrument console models the console and isolation system by a damped spring-mass oscillator with mass m = 2 kg, spring constant k = 8 N/m, and damping constant c = 1 N-s/m. To determine the system's response to external vibration, she assumes that the mass is initially stationary with the spring unstretched, and at t = 0 a force F(t) = 20 sin4t N is applied to the mass.

a) What is the amplitude of the steady-state solution?
b) What is the position of the mass as a function of time?



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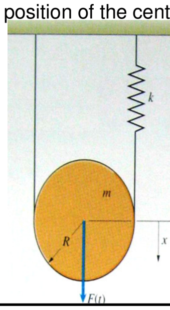
Example

The homogeneous disk has radius R = 2 ft and mass m = 4 slugs. The spring constant is k = 30 lb/ft. The disk is initially stationary in its equilibrium position, and at t = 0 a downward force $F(t) = 12 + 12t – 0.6t^2$ lb is applied to the center of the disk. Determine the position of the center of the disk as a function of time.



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