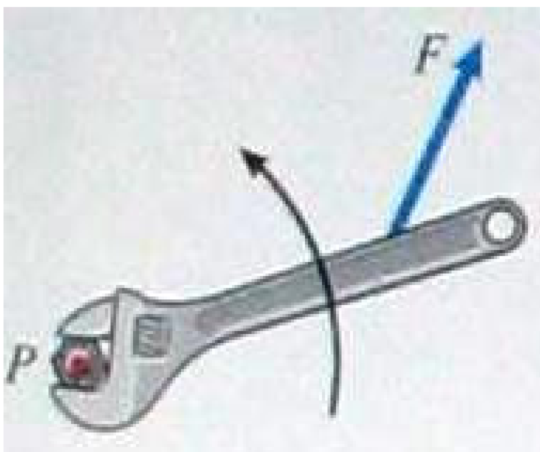
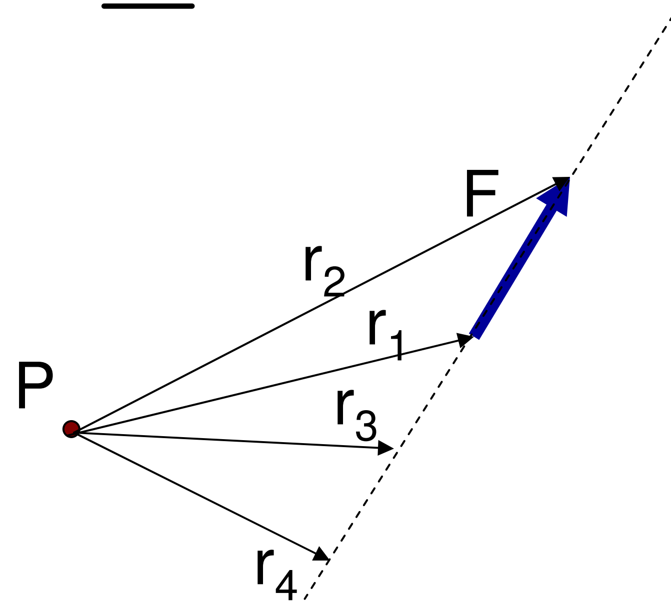
Moment: Linear Algebra vs. Trig

NOT $\underline{F} \times \underline{r}$!

$$M_p = \underline{r} \times \underline{F}$$

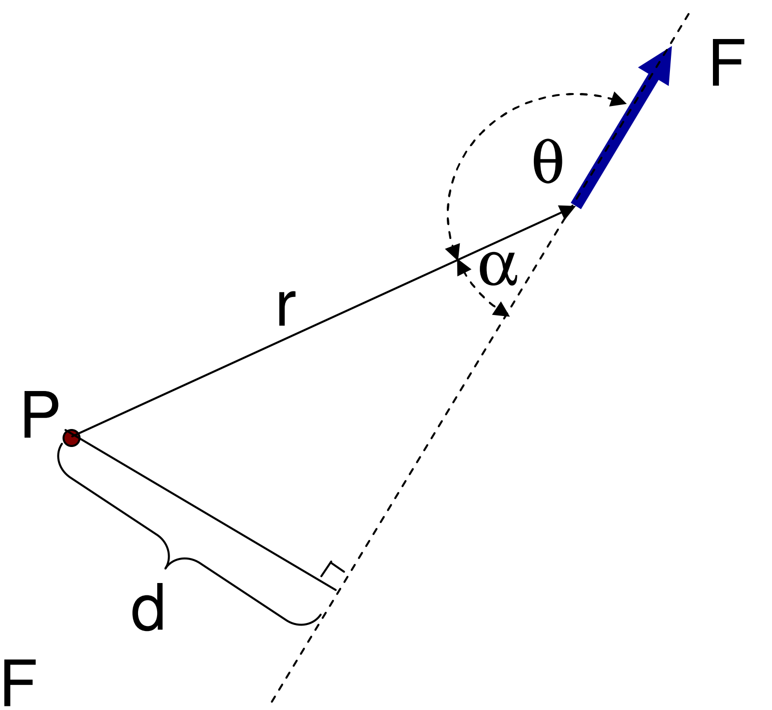




<u>If you use the Cross product</u> (not trig), any <u>r</u> that goes from point P to the line of action of force F is acceptable.
r does not have to be perpendicular to F.

Moment: Cross Product ➔ trig

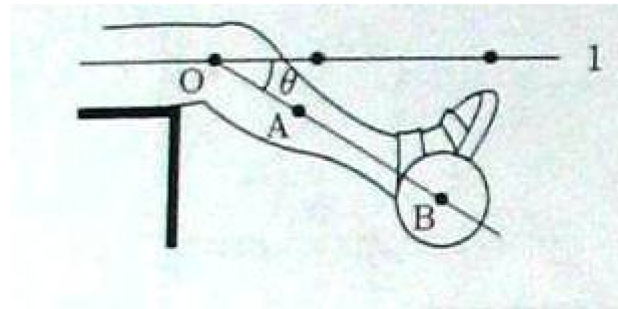
$$\underline{M}_p = \underline{r} \times \underline{F}$$
$$= |\underline{r}||\underline{F}|\sin\theta\hat{e}$$
$$= |\underline{r}||\underline{F}|\sin(180-\alpha)\hat{e}$$
$$= |\underline{r}||\underline{F}|\sin(\alpha)\hat{e}$$
$$= |\underline{r}|\sin(\alpha)|\underline{F}|\hat{e}$$
$$= d|\underline{F}|\hat{e}$$

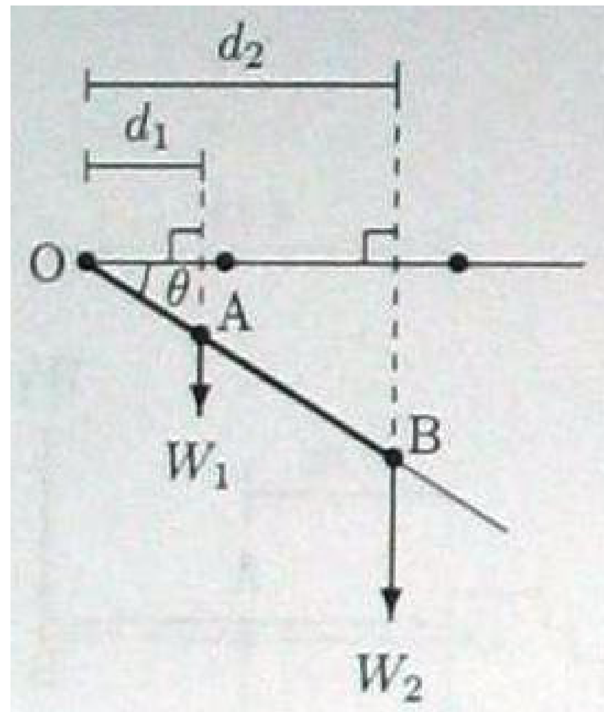


$M_p = dF$, where d is the perpendicular distance from P to F

Example: Calculate 2D Moment

Consider an athlete wearing a weight boot. The weight of the athlete's lower leg is W_1=50N and the weight of the boot is W_2 = 100 N. As measured from the knee joint at O, the center of mass (A) of the lower leg is located at a distance a=20 cm and the center of mass (B) of the weight boot is located at a distance b=50 cm.





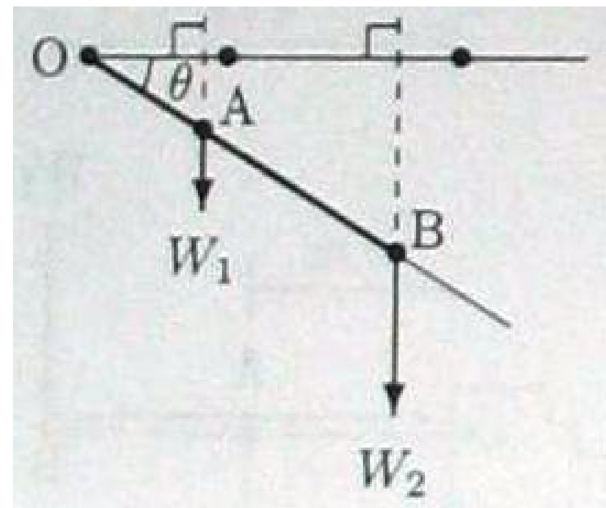
Determine the net moment generated about the knee joint when the lower leg is extended horizontally (position 1) and when the lower leg makes an angle of 30°

Solution, θ=0°

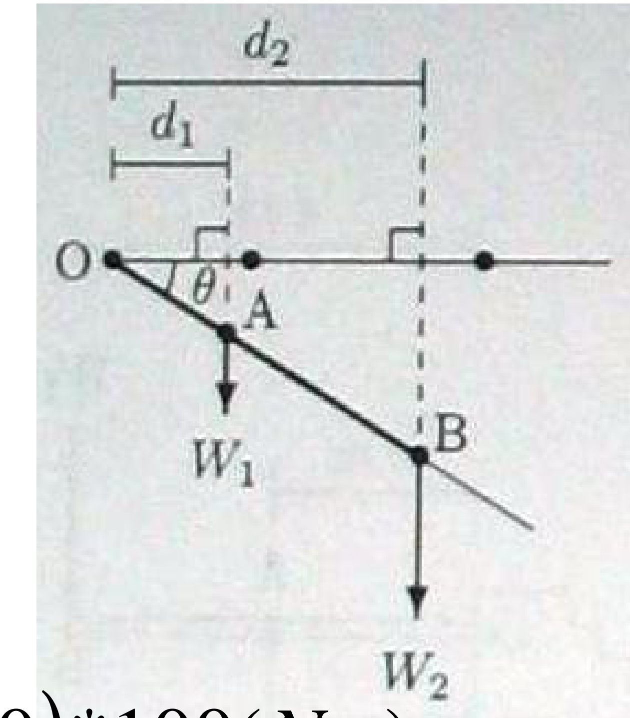
$$\circlearrowleft \sum M = -d_1W_1 - d_2W_2$$

$$\circlearrowleft \sum M = -0.2*50 - 0.5*100(Nm)$$

$$\circlearrowleft \sum M = -60Nm \quad ccw$$



Solution, θ=30°



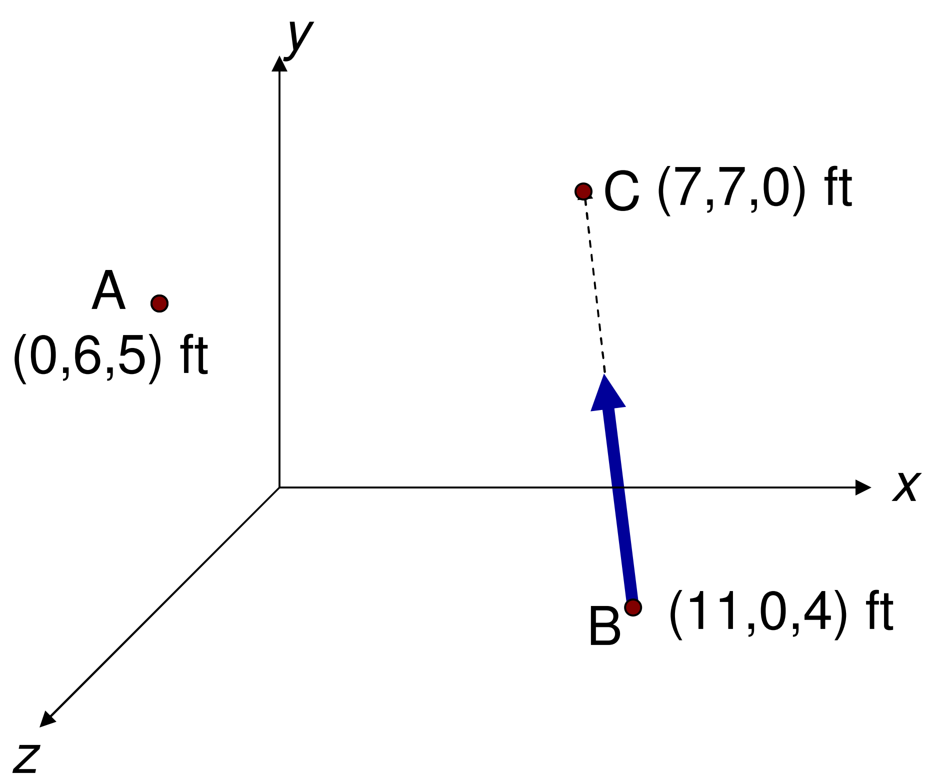
$$\circlearrowright \sum M = -d_1 W_1 - d_2 W_2$$

$$\circlearrowright \sum M = -0.2\cos(30)*50 - 0.5\cos(30)*100 (Nm)$$

$$\circlearrowright \sum M = -52 Nm \quad ccw$$

Example: Calculate 3D Moment

The line of action of the 90 N force passes through points B and C. What is the moment of F about point A?

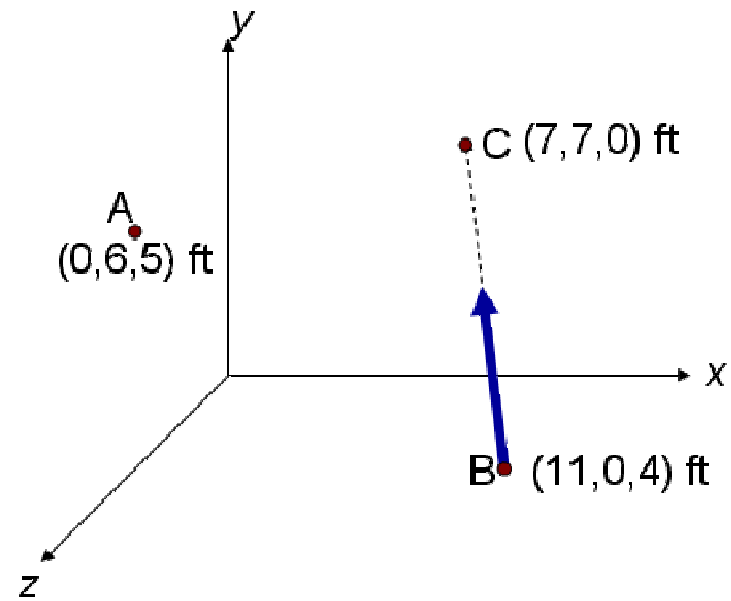


Solution

1) Determine Vector F

a) Find BC

$$\underline{BC} = (7-11)\hat{i} + (7-0)\hat{j} + (0-4)\hat{k} = -4\hat{i} + 7\hat{j} - 4\hat{k}\,(ft)$$



b) Find $\hat{e}_{BC}$

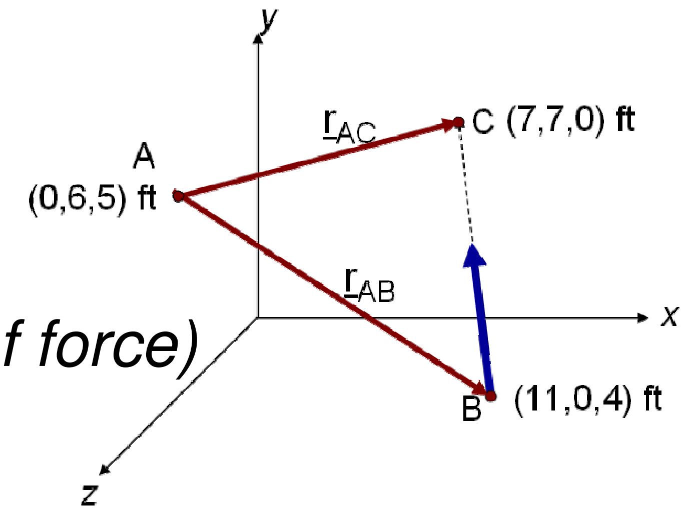
$$\left|\underline{BC}\right| = \sqrt{(-4)^2 + 7^2 + (-4)^2} = 9$$

$$\hat{e}_{BC} = -\frac{4}{9}\hat{i} + \frac{7}{9}\hat{j} - \frac{4}{9}\hat{k}$$

c) Scale $\hat{e}_{BC}$ by force magnitude to find F

$$\underline{F} = 90\hat{e}_{BC} = -40\hat{i} + 70\hat{j} - 40\hat{k}\,(lb)$$

Solution using $\underline{r}_{AB}$



2) Determine Vector $\underline{r}$
(point A to any point on line of action of force)

2 known points on the line of action are B and C.
To demonstrate that you can choose any point, we will use both $\underline{r}=\underline{AB}$ and $\underline{r}=\underline{BC}$

$$\underline{r}_{AB} = (11-0)\hat{i} + (0-6)\hat{j} + (4-5)\hat{k} = 11\hat{i} - 6\hat{j} - \hat{k}\,(ft)$$

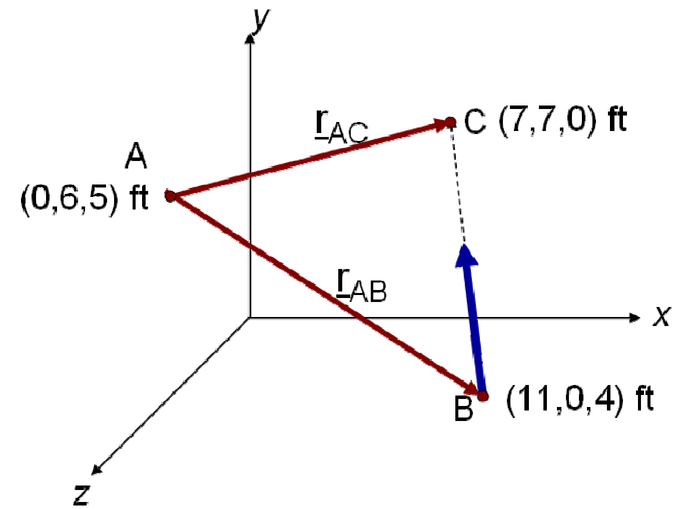
3) Evaluate $\underline{M} = \underline{r} \times \underline{F}$

$$M_A = \underline{r}_{AB} \times \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_{AB\,x} & r_{AB\,y} & r_{AB\,z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 11 & -6 & -1 \\ -40 & 70 & -40 \end{vmatrix}$$

$$= 310\hat{i} + 480\hat{j} + 530\hat{k}\,(ft-lb)$$

Solution using $\underline{r}_{AC}$

2) Determine Vector $\underline{r}$



$$\underline{r}_{AC} = (7-0)\hat{i} + (7-6)\hat{j} + (0-5)\hat{k} = 7\hat{i} + \hat{j} - 5\hat{k}\,(ft)$$

3) Evaluate $\underline{M} = \underline{r} \times \underline{F}$

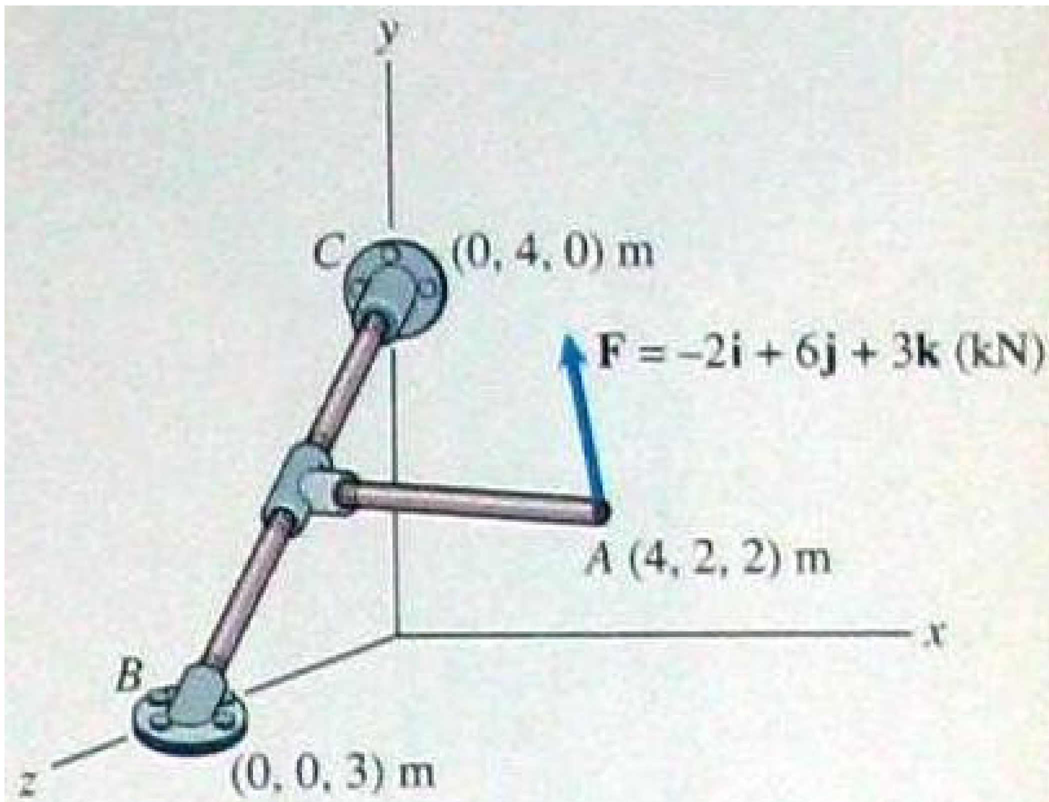
$$M_A = \underline{r}_{AC} \times \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r_{AC\,x} & r_{AC\,y} & r_{AC\,z} \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & -1 & -5 \\ -40 & 70 & -40 \end{vmatrix}$$

$$= 310\hat{i} + 480\hat{j} + 530\hat{k}\,(ft-lb)$$

Same answer

Example: Moment about an Axis

- What is the moment of F about bar BC?



Solution

1) Determine Vector r
   (any point on BC to any point on line of action of force)

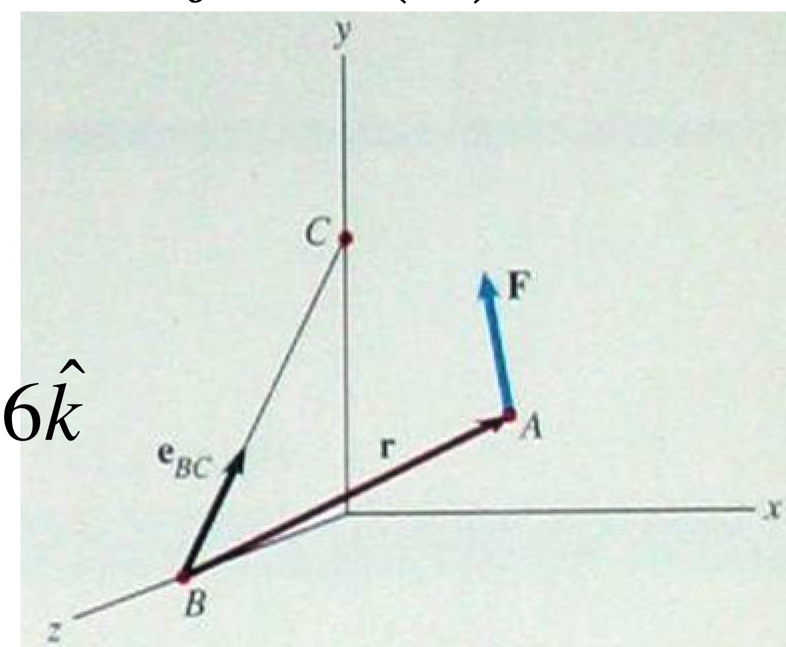
$$r = BA = (4-0)\hat{i} + (2-0)\hat{j} + (2-3)\hat{k} = 4\hat{i} + 2\hat{j} - \hat{k}(m)$$

2) Determine Vector ê along BC

$$\underline{BC} = (0-0)\hat{i} + (4-0)\hat{j} + (0-3)\hat{k} = 4\hat{j} - 3\hat{k}(m)$$

$$|\underline{BC}| = \sqrt{0^2 + 4^2 + (-3)^2} = 5$$

$$\hat{e}_{BC} = \frac{\underline{BC}}{|\underline{BC}|} = \frac{4\hat{j} - 3\hat{k}}{5} = 0.8\hat{j} - 0.6\hat{k}$$



Solution *(continued)*

3) Evaluate M_L using mixed triple product

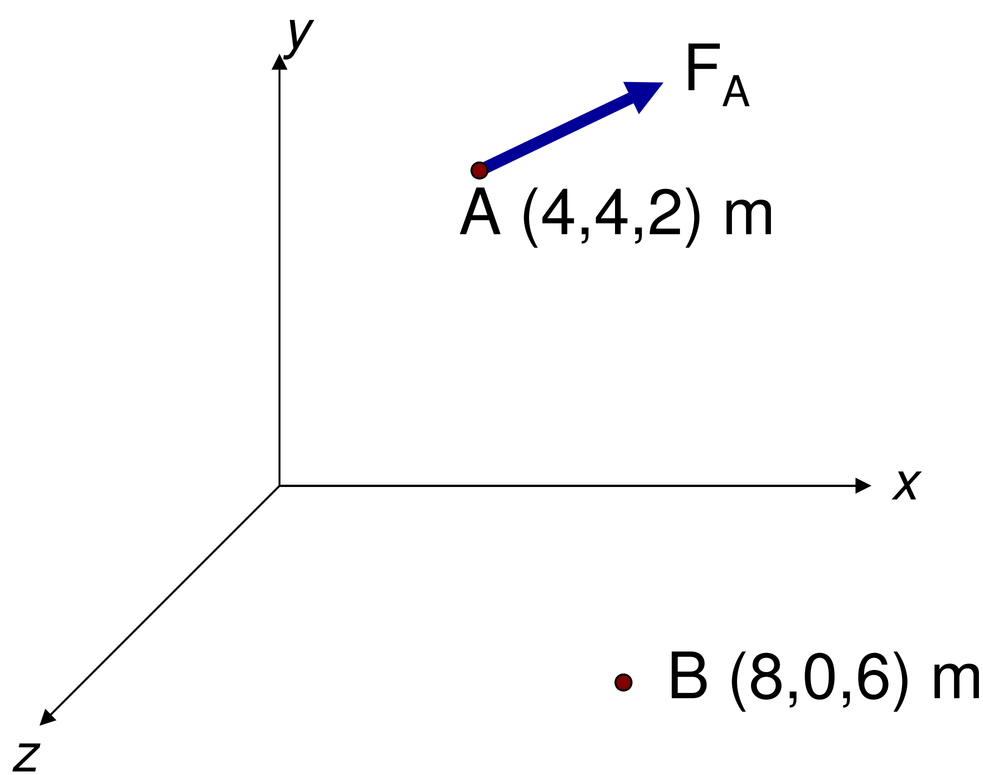
$$\hat{e}_{BC} \cdot \left(\underline{r} \times \underline{F}\right) = \begin{vmatrix} \hat{e}_{BC\,x} & \hat{e}_{BC\,y} & \hat{e}_{BC\,z} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix} = \begin{vmatrix} 0 & .8 & .6 \\ 4 & 2 & -1 \\ -2 & 6 & 3 \end{vmatrix} = -24.8kNm$$

$$\underline{M}_{BC} = \left[\hat{e}_{BC} \cdot \left(\underline{r} \times \underline{F}\right)\right]\hat{e}_{BC} = -24.8\hat{e}_{BC}\left(kNm\right) = -19.84\hat{j} + 14.88\hat{k}$$

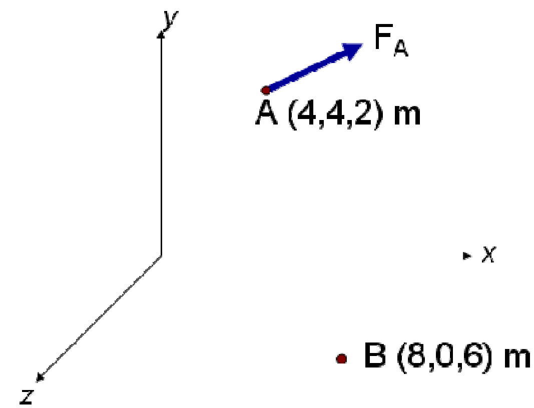
Example: Moving a Force

Force F_A = $10\hat{i} + 4\hat{j} - 3\hat{k}(N)$

acting at A. Represent it by a force acting at B and a couple



Solution



1) Sum of Forces must be equal:

$$\left(\sum F\right)_2 = \left(\sum F\right)_1 : F = F_A = 10\hat{i} + 4\hat{j} - 3\hat{k}\,(N)$$

2) Sum of Moments about any point must be equal:

$$r_{BA} = (4-8)\hat{i} + (4-0)\hat{j} + (2-6)\hat{k} = -4\hat{i} + 4\hat{j} + -4\hat{k}$$

System 1 Moment about B:

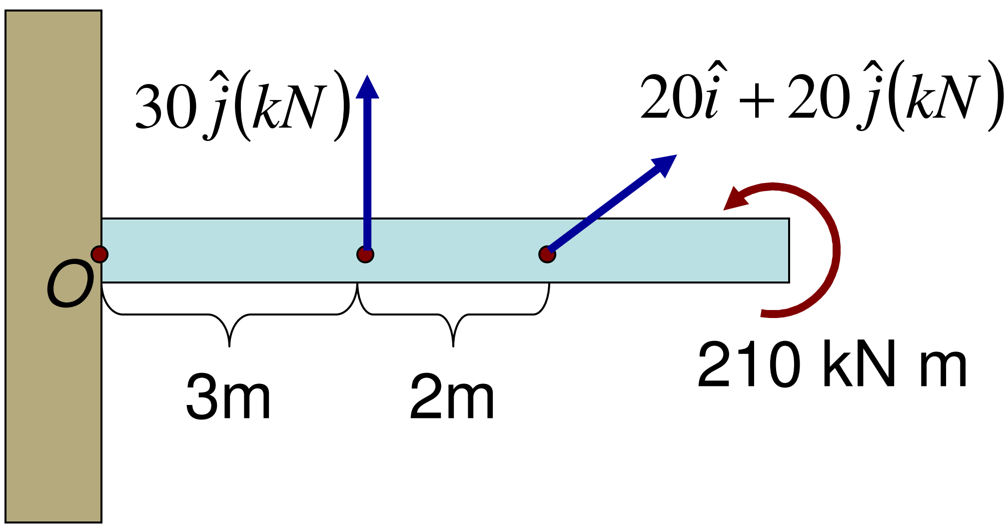
$$r_{BA} \times F_A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & 4 & -4 \\ 10 & 4 & -3 \end{vmatrix} = 4\hat{i} - 52\hat{j} - 56\hat{k}\,(Nm)$$

$$(M_B)_2 = (M_B)_1$$

$$M = 4\hat{i} - 52\hat{j} - 56\hat{k}\,(Nm)$$

Example: Equivalent Systems

Represent this system by:
a) a single force acting at *O*
b) by a single force



Solution (a)

1) Sums of Forces must be equal

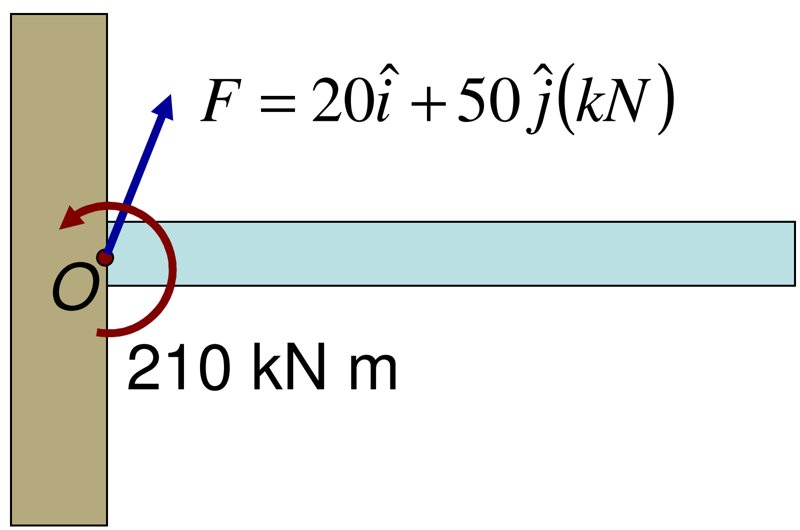
$$\left(\sum F\right)_2 = \left(\sum F\right)_1 : \ F = 30\hat{j} + \left(20\hat{i} + 20\hat{j}\right) = 20\hat{i} + 50\hat{j}(kN)$$

2) Sums of Moments must be equal

$$\left(M_O\right)_2 = \left(M_O\right)_1$$

$$M = \left(30kN\right)\left(3m\right) + \left(20kN\right)\left(5m\right) + 210kN - m$$

M=400 kN-m



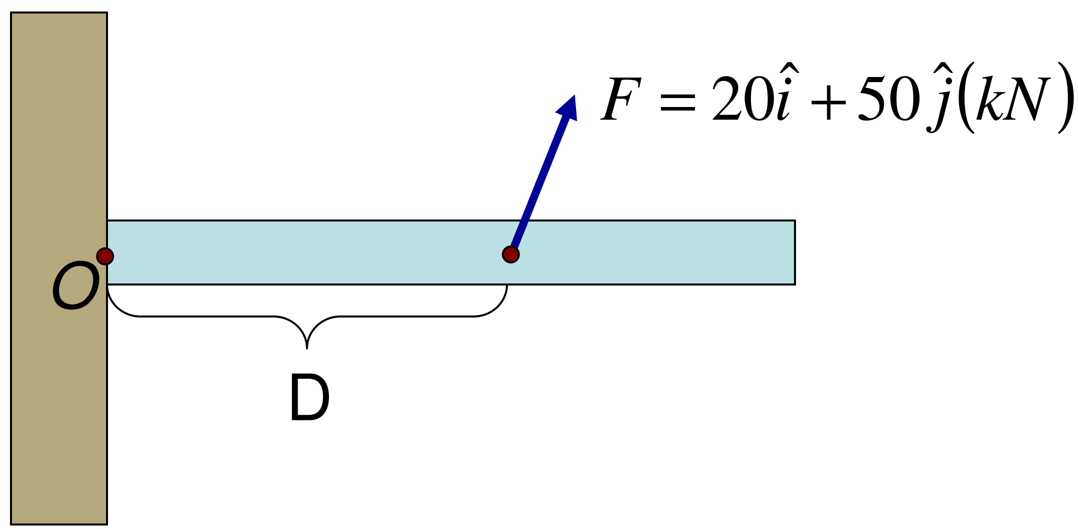
Solution (b)

2) Sums of Moments must be equal

$$(M_O)_3 = (M_O)_2$$

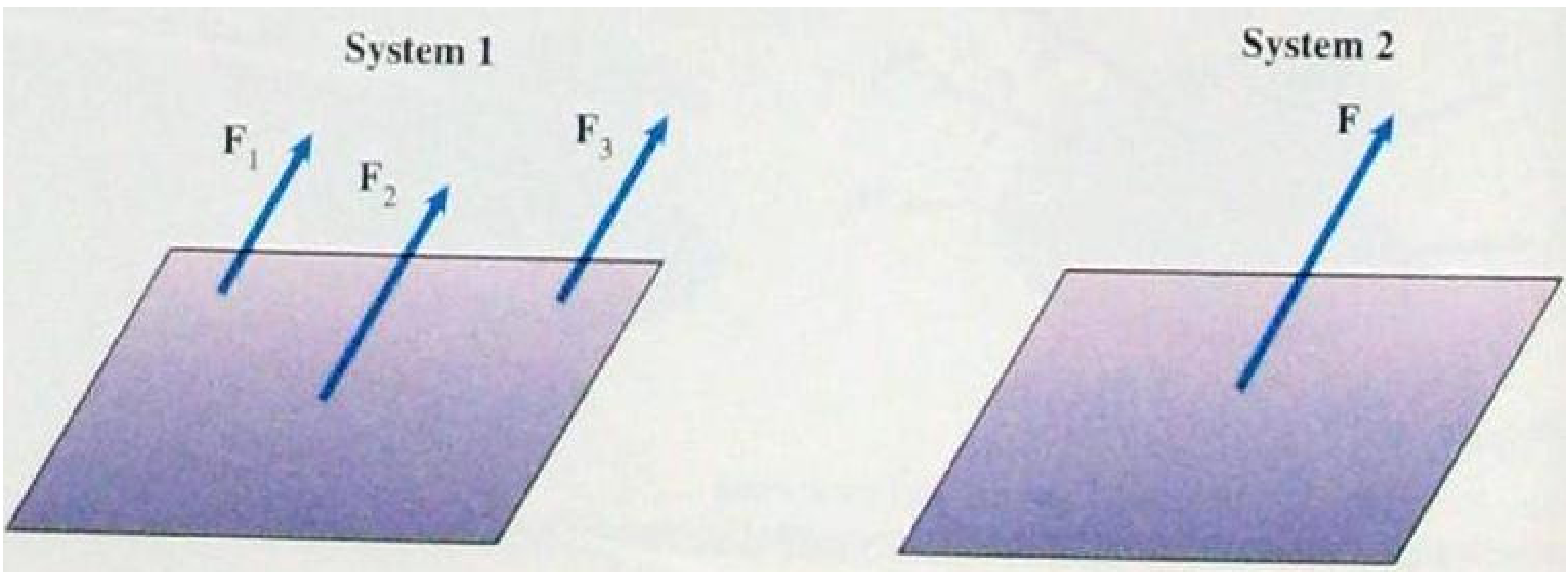
(50 kN)D = 400 kN-m

D = 8 m



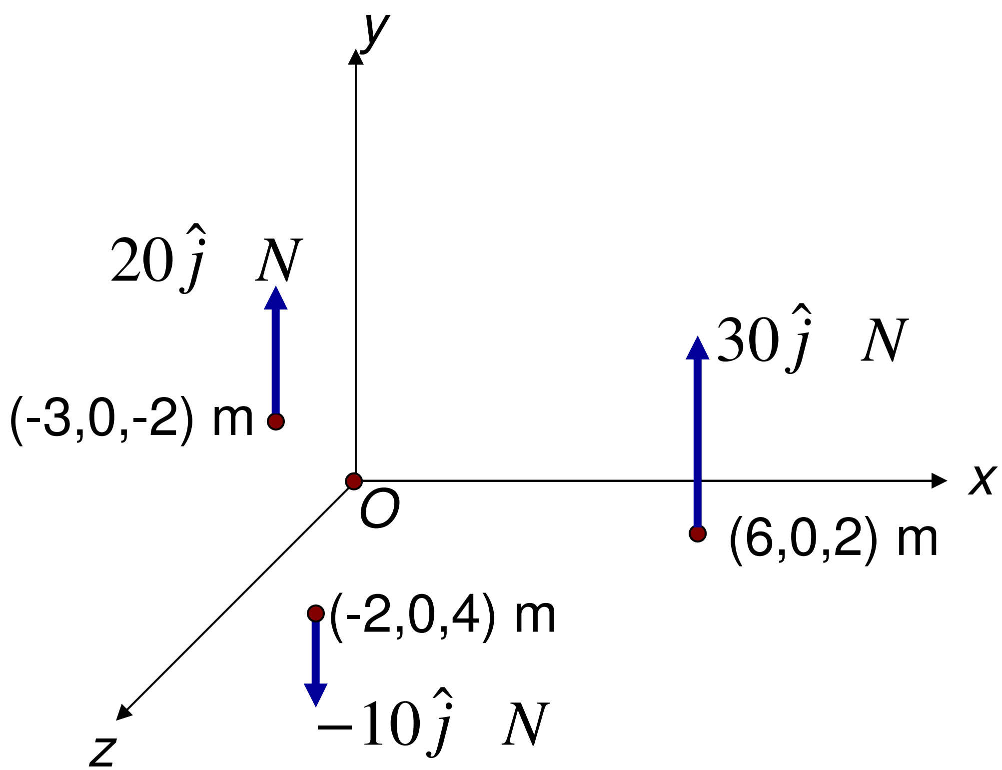
Common Situation: Parallel Forces

The resulting force is the net sum of forces
Placement is critical – sum of moments must be equal



Example: Parallel Forces

Simplify the system of forces to a single force



Solution

1) Sums of Forces must be equal

$$\left(\sum F\right)_2 = \left(\sum F\right)_1 : \; F = 30\hat{j} + 20\hat{j} - 10\hat{j} = 40\hat{j}(N)$$

2) Sums of Moments must be equal

Let coordinates of point P be (x,y,z)

Sum moments about origin

$$\left(M_B\right)_2 = \left(M_B\right)_1$$

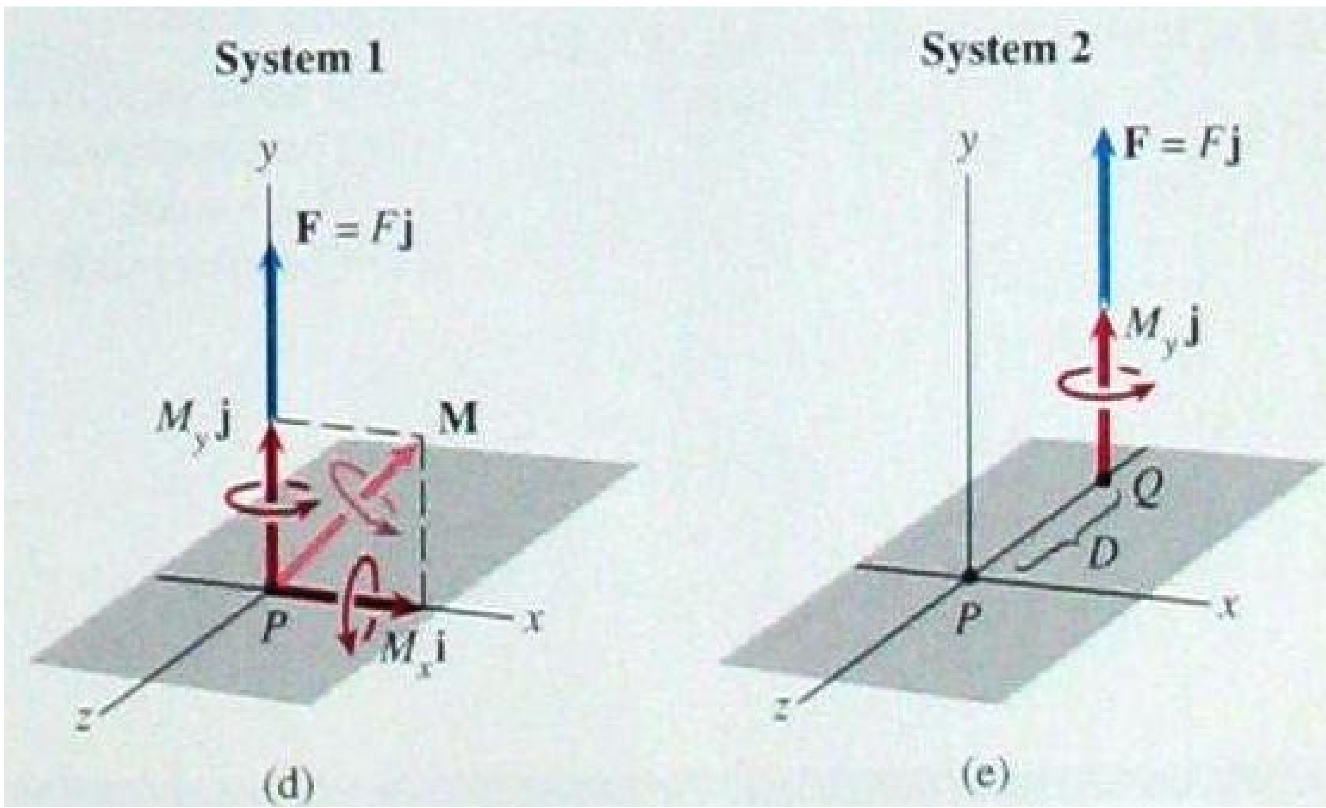
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 0 & 40 & 0 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 6 & 0 & 2 \\ 0 & 30 & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & 4 \\ 0 & -10 & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 0 & -2 \\ 0 & 20 & 0 \end{vmatrix}$$

$$(20 + 40z)\hat{i} + (100 - 40x)\hat{k} = 0$$

x = 2.5m, z = -0.5 m, y can be any position

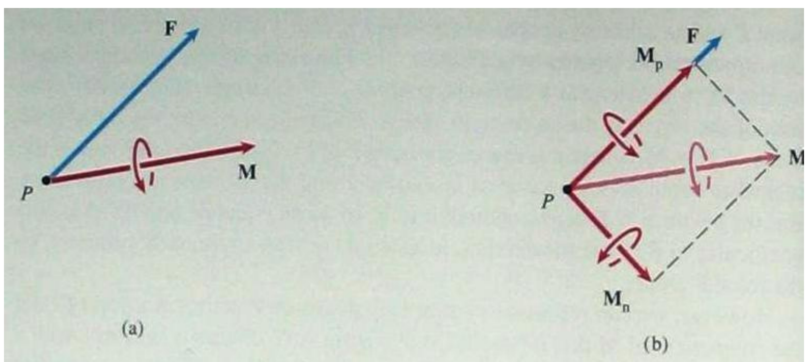
Simplest generic system: Wrench

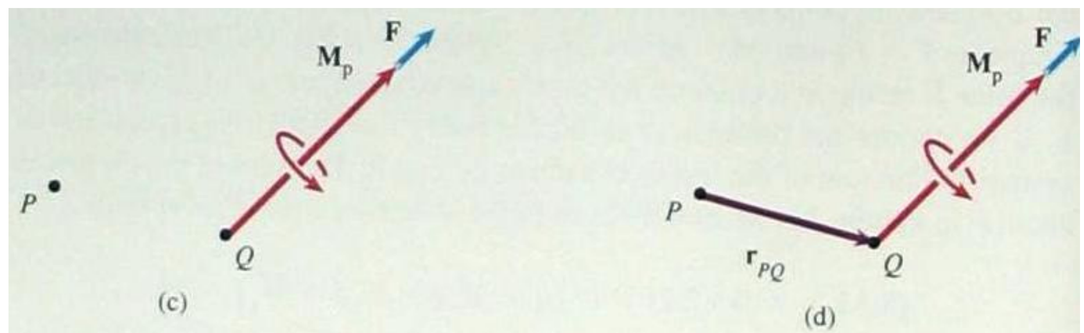
Consists of a force and a moment parallel to the force



Calculating the equivalent wrench

1) First find the equivalent system (force + couple)
2) Determine the components of M parallel and normal to F
3) Wrench consists of force F and moment M_P, acting at point Q.
4) Choose point Q so that $\underline{r}_{PQ} \times \underline{F} = \underline{M}_n$

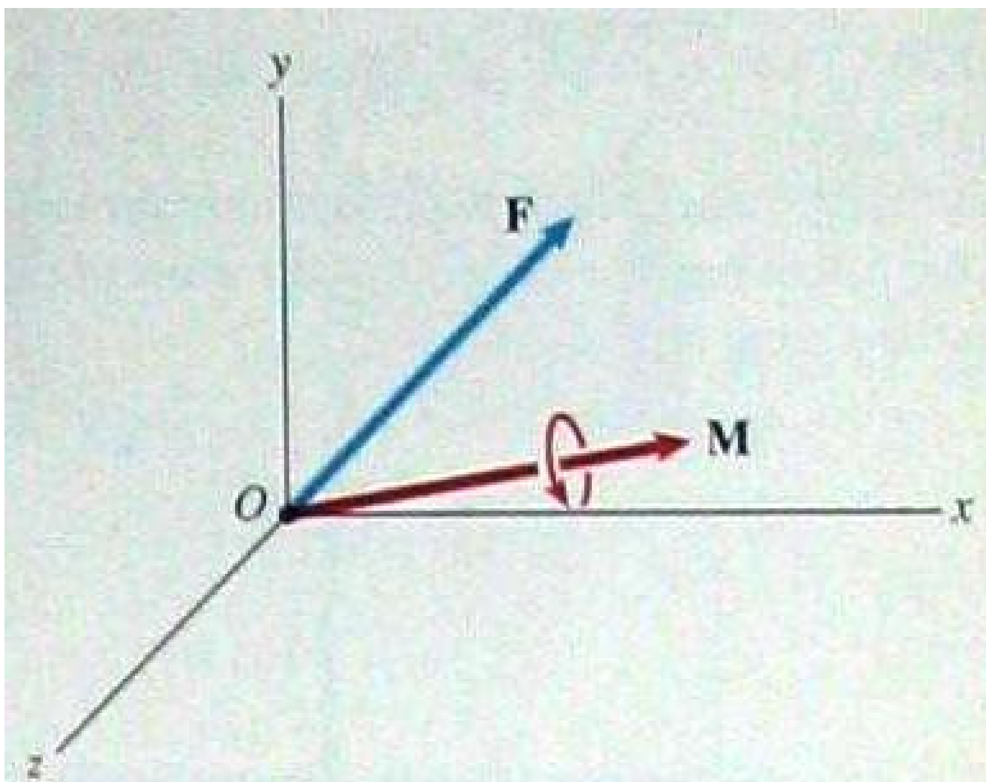




Wrench Example

$$\underline{F} = 3\hat{i} + 6\hat{j} + 2\hat{k} \quad (N)$$

$$\underline{M} = 12\hat{i} + 4\hat{j} + 6\hat{k} \quad (Nm)$$



Solution

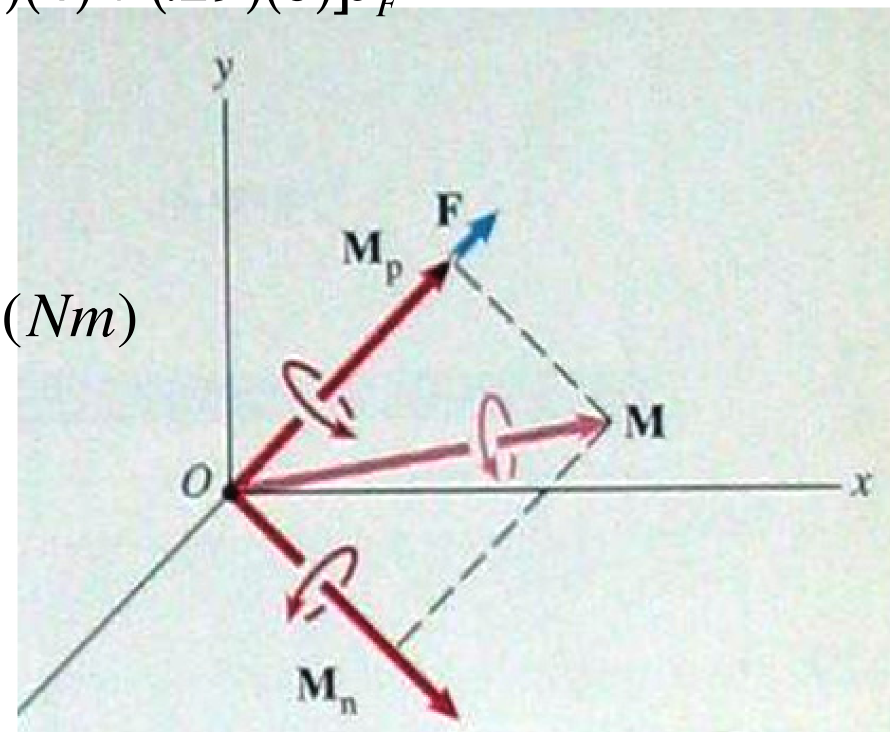
1) We were given an equivalent system
2) Determine components of M parallel and normal to F

$$\hat{e}_F = \frac{\underline{F}}{|\underline{F}|} = \frac{3\hat{i} + 6\hat{j} + 2\hat{k}}{\sqrt{3^2 + 6^2 + 2^2}} = 0.43\hat{i} + 0.86\hat{j} + 0.29\hat{k}$$

$$\underline{M}_p = (\hat{e}_F \cdot \underline{M})\hat{e}_F = [(.43)(12) + (.86)(4) + (.29)(6)]\hat{e}_F$$

$$\underline{M}_p = 4.4\hat{i} + 8.8\hat{j} + 2.9\hat{k}(Nm)$$

$$\underline{M}_n = \underline{M} - \underline{M}_p = 7.6\hat{i} - 4.8\hat{j} + 3.1\hat{k}(Nm)$$



Solution (continued)

3) Wrench consists of force $\underline{F}$ and moment $\underline{M}_P$, acting at point P.

4) Choose location of P so that $\underline{r}_{OP}$ x $\underline{F}$ = $\underline{M}_n$

Let coordinates of P be (x,0,z)

$$\underline{r}_{OP} \times \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & 0 & z \\ 3 & 6 & 2 \end{vmatrix} = -6z\hat{i} - (2x - 3z)\hat{j} + 6x\hat{k}$$

Set $\underline{r}_{OP}$ x $\underline{F}$ = $\underline{M}_n$:

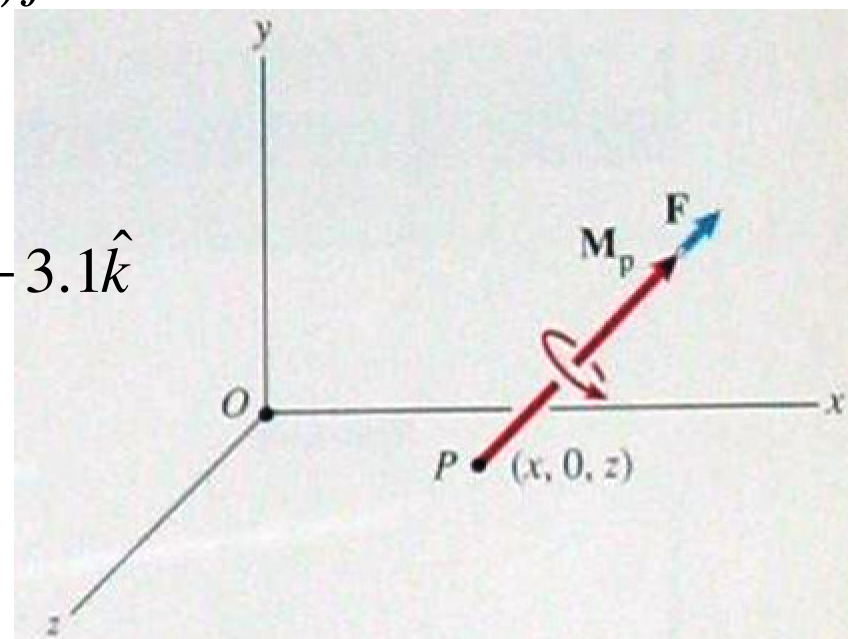
$$-6z\hat{i} - (2x - 3z)\hat{j} + 6x\hat{k} = 7.6\hat{i} - 4.8\hat{j} + 3.1\hat{k}$$

Each component must be equal:

$$-6z = 7.6$$

$$-2x + 3z = -4.8$$

$$6x = 3.1$$



Coordinates of P are x=0.51m, y = 0 *(predetermined)*, z=-1.27m